Problem 9.2 of Shuler & Kargi. Two fermentors in series.
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Cell growth parameters:

$\mu_m := 0.3\ hr^{-1}$ $K_s := 0.1\ \frac{g}{liter}$ $\mu_1(s) := \frac{\mu_m \cdot s}{K_s + s}$ $\mu_2(s) := 0$ $Y_s := 0.4\ \frac{g\_cell}{g\_S}$

Product formation parameters:

$Y_p := 0.6\ \frac{g\_P}{g\_S}$ $q_p := 0.02\ \frac{g\_P}{g\_cell \cdot hr}$

Substrate feed rate:

$F := 100\ \frac{liter}{hr}$ $s_f := 5\ \frac{gm}{liter}$ $D_{washout} := \mu_1(s_f)$

Steady-state equations for the first fermentor. $V_1 := 500\ liter$ $D_1 := \frac{F}{V_1} = 0.2\ hr^{-1}$

$x_1 := 2$ $s_1 := 0$ ... initial guess

Given

dx_1/dt $0 = \mu_1(s_1) \cdot x_1 - D_1 \cdot x_1$

ds_1/dt $0 = D_1 \cdot (s_f - s_1) - \frac{1}{Y_s} \cdot \mu_1(s_1) \cdot x_1$

$xs1(D_1) := Find(x_1, s_1)$ $x_1(D_1) := xs1(D_1)_0$ $s_1(D_1) := xs1(D_1)_1$ $xs1(D_1) = \begin{pmatrix} 1.92 \\ 0.2 \end{pmatrix}$

Dynamic equations for the second fermentor. $V_2 := 300\ liter$ $D_2 := \frac{F}{V_2} = 0.333\ hr^{-1}$

$x_2 := 1$ $s_2 := 0$ $p_2 := 0$ ... initial guess

Given

dx_2/dt $0 = D_2 \cdot (x_1(D_1) - x_2) + \mu_2(s_2) \cdot x_2$

ds_2/dt $0 = D_2 \cdot (s_1(D_1) - s_2) - \frac{1}{Y_s} \cdot \mu_2(s_2) \cdot x_2 - \frac{1}{Y_p} \cdot q_p \cdot x_2$

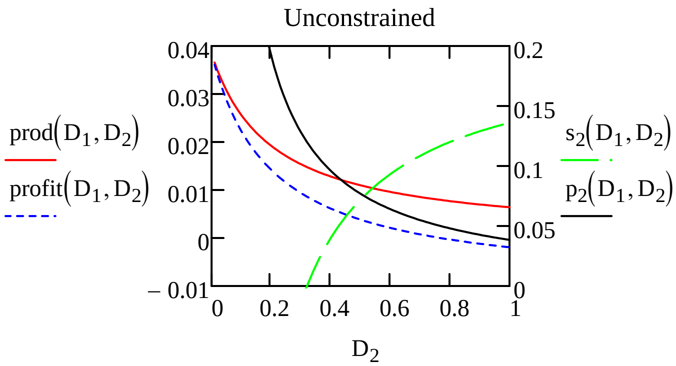
ds_2/dt $0 = q_p \cdot x_2 - D_2 \cdot p_2$

$xsp2(D_1, D_2) := Find(x_2, s_2, p_2)$ $s_2(D_1, D_2) := xsp2(D_1, D_2)_1$ $p_2(D_1, D_2) := xsp2(D_1, D_2)_2$

$prod(D_1, D_2) := \frac{p_2(D_1, D_2)}{\frac{1}{D_1} + \frac{1}{D_2}}$ $profit(D_1, D_2) := \frac{p_2(D_1, D_2) - 0.01 \cdot s_f}{\frac{1}{D_1} + \frac{1}{D_2}}$ The value of s is 0.01X the value of p.

$prod(D_1, D_2) = 0.014$ $xsp2(D_1, D_2) = \begin{pmatrix} 1.92 \\ 8 \times 10^{-3} \\ 0.115 \end{pmatrix}$

$D_2 := 0.01, 0.02 .. 1$



Product concentration and product productivity continue to rise as $D_2 \rightarrow 0$. However, this is only because we failed to impose the constraint $0 \le s_2$. Notice $0 \le s_2$ for small values of D_2.

Optimization without constraint → high product productivity, but unreal because of negative D_2.

$D_1 := 0.1 \quad D_2 := 0.3$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} := \text{Maximize}(\text{prod}, D_1, D_2) = \begin{pmatrix} 0.049 \\ -0.049 \end{pmatrix} \qquad \text{prod}(D_1, D_2) = 2.68 \times 10^4$$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} := \text{Maximize}(\text{profit}, D_1, D_2) = \begin{pmatrix} 0.049 \\ -0.049 \end{pmatrix} \qquad \text{profit}(D_1, D_2) = 2.843 \times 10^4$$

$s_2(D_1, D_2) = 1.385$

Optimization subject to inequality constraint $0 \le s_2$.

$D_1 := 0.1 \quad D_2 := 0.3 \qquad \text{Given} \qquad 0 \le s_2(D_1, D_2)$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} := \text{Maximize}(\text{prod}, D_1, D_2) = \begin{pmatrix} 0.271 \\ 0.057 \end{pmatrix}$$

$\text{prod}(D_1, D_2) = 0.027 \qquad s_2(D_1, D_2) = -0$

Another way is to define "prod" only for $0 \le s_2$.

$$\text{prod}(D_1, D_2) := (0 \le s_2(D_1, D_2)) \cdot \frac{p_2(D_1, D_2)}{\frac{1}{D_1} + \frac{1}{D_2}}$$

$$\text{profit}(D_1, D_2) := (0 \le s_2(D_1, D_2)) \cdot \frac{p_2(D_1, D_2) - 0.01 \cdot s_f}{\frac{1}{D_1} + \frac{1}{D_2}}$$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} := \text{Maximize}(\text{prod}, D_1, D_2) = \begin{pmatrix} 0.27 \\ 0.06 \end{pmatrix}$$

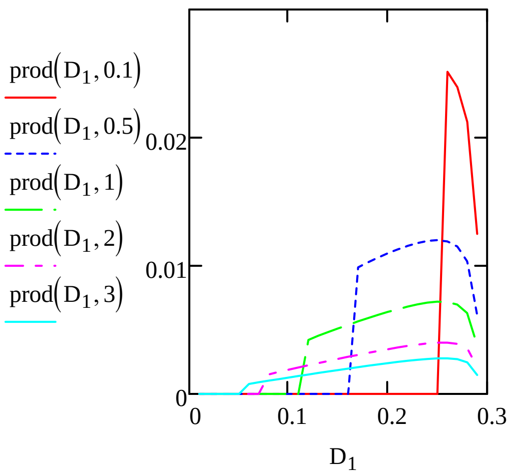
$\text{prod}(D_1, D_2) = 0.027 \qquad s_2(D_1, D_2) = 0$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} := \text{Maximize}(\text{profit}, D_1, D_2) = \begin{pmatrix} 0.27 \\ 0.06 \end{pmatrix}$$

Optimization finds a balance in allocating substrate for biomass production in the first fermentor and substrate for production production in the second fermentor. Substrate is all utilized at the end. The optimum is right at the constraint boundary.

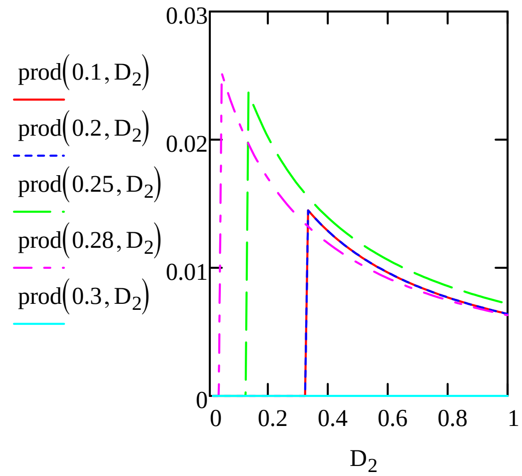
Effect of changing D_1 at fixed D_2.

$D_1 := 0.01, 0.02 .. D_{washout}$



Effect of changing D_2 at fixed D_1.

$D_2 := 0.01, 0.02 .. 1$

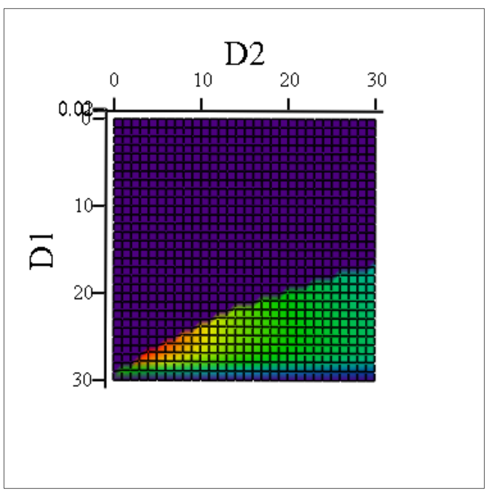


3-D plot

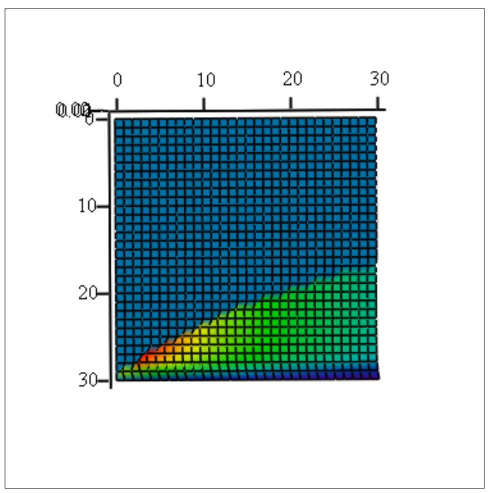
$ni := 30 \qquad i := 0 .. ni \qquad \Delta D_1 := \frac{0.3}{ni} \qquad D_{1_i} := i \cdot \Delta D_1 + 0.00001$

$nj := 30 \qquad j := 0 .. nj \qquad \Delta D_2 := \frac{0.5}{nj} \qquad D_{2_j} := j \cdot \Delta D_2 + 0.00001$

$PROD_{i,j} := prod\left(D_{1_i}, D_{2_j}\right) \qquad PROFIT_{i,j} := profit\left(D_{1_i}, D_{2_j}\right)$



PROD



PROFIT

$\max(PROD) = 0.026$

$index := match(\max(PROD), PROD)_0$

$\overrightarrow{\left[index \cdot \begin{pmatrix} \Delta D_1 \\ \Delta D_2 \end{pmatrix}\right]} = \begin{pmatrix} 0.27 \\ 0.067 \end{pmatrix}$ D_1 & D_2